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Corner Transfer Matrices of the Chiral Potts Model

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We present some symmetry and factorization relations satisfied by the corner transfer matrices (CTMs) of the chiral Potts model. We show how the single-spin expectation values can be expressed in terms of the CTMs, and in terms of the related boost operator. Low-temperature calculations lead naturally to the variables that uniformize the Boltzmann weights of the model.

KEY WORDS: Statistical mechanics; integrable models; lattice statistics; corner transfer matrices; Chiral potts model.

1. INTRODUCTION

The "chiral Potts" model is a special case of the Z_N-symmetric, chirally asymmetric N-state model that satisfies the star–triangle relation.[1,2] Because of this, we expect it to be solvable in the same sense that the eight-vertex model[3] is solvable. In particular, one should be able to obtain the free energy, correlation lengths, interfacial tensions, and single-spin expectation values, all in the thermodynamic limit of an infinite lattice. In previously solved models, the single-spin expectation values have been obtained quite easily by a route that uses corner transfer matrices (CTMs). However, the method depends on the "difference property," which is lacking in the chiral Potts model.

It is not yet clear how to overcome this difficulty, but here we follow the route as far as we can. We show that the chiral Potts CTMs can be expressed in terms of a matrix function A_p of a single rapidity p, that satisfies a "quasiperiodicity" relation involving a rapidity-independent matrix M. The single-spin expectation values $\langle Z_0^j \rangle$ can be obtained from M, most easily from the diagonalized form of M. Also, A_p and M can be

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related to a "boost operator" $\mathscr{B}_p$. We diagonalize $\mathscr{B}_p$ and M explicitly for the $N=2$ Ising case; and for the general-N case in the low-temperature limit; and hence obtain $\langle Z_0^j \rangle$ for these cases.

These calculations automatically introduce the arguments of the hyperelliptic theta functions that provide a uniformizing substitution for the model.[4]

2. DEFINITIONS

We define the chiral Potts model in the usual way.[5,6] Consider the square lattice $\mathscr{L}$, drawn diagonally as in Fig. 1. At each site i there is a spin σ_i, which takes values $0,..., N-1$. There is an associated lattice $\mathscr{L}'$ denoted by dotted lines, such that each edge of $\mathscr{L}$ passes through a vertex of $\mathscr{L}'$.

Let k be a given real constant, $0<k<1$, $k'=(1-k^2)^{1/2}$, $\gamma=\operatorname{arcosh}(1/k)$, and let $\omega=\exp(2\pi i/N)$. Let $p=\{x_p, y_p, t_p, \lambda_p, \mu_p\}$ be a set of complex numbers ("p-variables"), related by

$$x_p^N + y_p^N = k(1+x_p^N y_p^N), \qquad kx_p^N = 1-k'\lambda_p^{-1}, \qquad ky_p^N = 1-k'\lambda_p$$
$$x_p y_p = t_p, \qquad \lambda_p = \mu_p^N \tag{1}$$

Once N, k, and any one of the "p-variables" $x_p,..., \mu_p$ are given, the rest are determined, to within a finite number of discrete choices of Nth roots and solutions of quadratic equations. In terms of the a_p, b_p, c_p, d_p of ref. 1, $x_p = a_p/d_p$, $y_p = b_p/c_p$, $\mu_p = d_p/c_p$. We refer to p as a "rapidity."

Similarly, define "q-variables" $q=\{x_q, y_q, t_q, \lambda_q, \mu_q\}$. To each vertical (horizontal) dotted line of Fig. 1 assign a rapidity p (q). In general they may be different for different lines. Then on a SW → NE edge (i, j) of $\mathscr{L}$

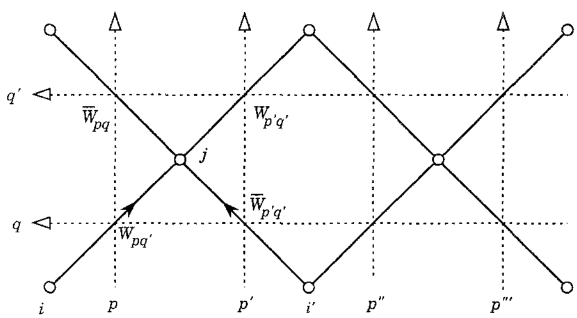

Fig. 1. The square lattice $\mathscr{L}$ (solid lines) and the associated rapidity-line lattice $\mathscr{L}'$ (broken lines).

(with j above i), the spins σ_i, σ_j interact with Boltzmann weight $W_{pq}(\sigma_i - \sigma_j)$, where

$$W_{pq}(n) = (\mu_p/\mu_q)^n \prod_{j=1}^{n} (y_q - \omega^j x_p)/(y_p - \omega^j x_q) \tag{2}$$

Similarly, on SE → NW edges the spins interact with Boltzmann weight $\bar{W}_{pq}(\sigma_i - \sigma_j)$, where

$$\bar{W}_{pq}(n) = (\mu_p \mu_q)^n \prod_{j=1}^{n} (\omega x_p - \omega^j x_q)/(y_q - \omega^j y_p) \tag{3}$$

Here we normalize so that $W_{pq}(0) = \bar{W}_{pq}(0) = 1$. The weights satisfy the periodicity relations $W_{pq}(n+N) = W_{pq}(n)$, $\bar{W}_{pq}(n+N) = \bar{W}_{pq}(n)$.

Now define the corner transfer matrix as in ref. 7. Fix the spins at the boundary sites of $\mathscr{L}$ to have value 0, and divide $\mathscr{L}$ into four quadrants (corners) as in Fig. 2, meeting at a central site 0. Let $\sigma = \{\sigma_0, ..., \sigma_{m-1}\}$

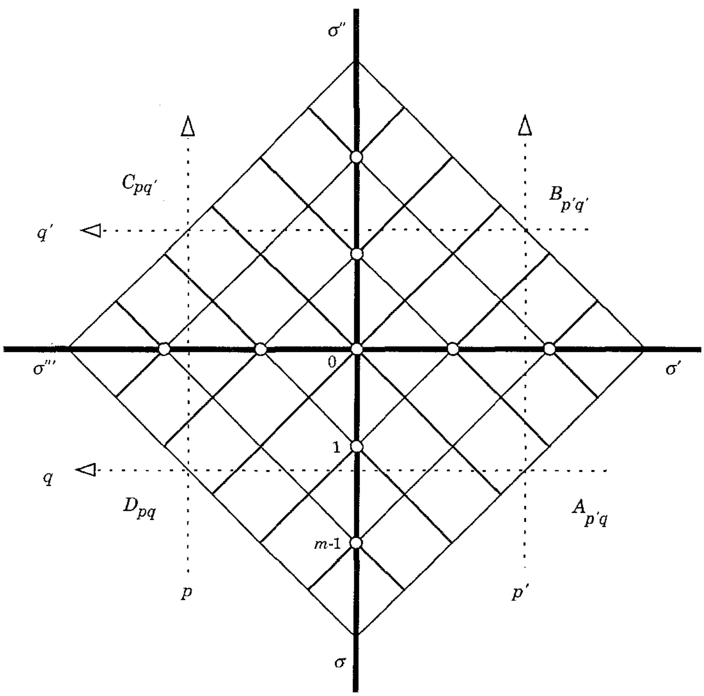

Fig. 2. The division of $\mathscr{L}$ into four quadrants, with corner transfer matrices A, B, C, D. Here $m = 3$.

denote the m free spins on the vertical half-line below 0 (including the center spin σ_0). Similarly, let σ', σ'', σ''' be the sets of m spins on the other three half-lines radiating from site 0, as shown in Fig. 2. Let $\tilde{A}_{\sigma\sigma'}$ be the product of the Boltzmann weights of all edges of $\mathscr{L}$ in the lower-right quadrant, summed over all spins *inside* that quadrant. Define $\tilde{B}_{\sigma'\sigma''}$, $\tilde{C}_{\sigma''\sigma'''}$, $\tilde{D}_{\sigma'''\sigma}$ similarly for the other three quadrants. Then the partition function Ξ is

$$\Xi = \sum_{\sigma} \sum_{\sigma'} \sum_{\sigma''} \sum_{\sigma'''} \tilde{A}_{\sigma\sigma'} \tilde{B}_{\sigma'\sigma''} \tilde{C}_{\sigma''\sigma'''} \tilde{D}_{\sigma'''\sigma} \tag{4}$$

Similarly, the average value of $\omega^{j\sigma_0}$ is

$$\langle \omega^{j\sigma_0} \rangle = \Xi^{-1} \sum_{\sigma} \sum_{\sigma'} \sum_{\sigma''} \sum_{\sigma'''} \omega^{j\sigma_0} \tilde{A}_{\sigma\sigma'} \tilde{B}_{\sigma'\sigma''} \tilde{C}_{\sigma''\sigma'''} \tilde{D}_{\sigma'''\sigma} \tag{5}$$

As in ref. 8, let $X_0,..., X_{m-1}$; $Z_0,..., Z_{m-1}$ be the N^m by N^m matrices with entries (for $r = 0,..., m-1$)

$$\begin{aligned} (X_r)_{\sigma\sigma'} &= \delta(\sigma_r, \sigma'_r + 1) \prod_{0 \leqslant i \leqslant m-1, i \neq r} \delta(\sigma_i, \sigma'_i) \\ (Z_r)_{\sigma\sigma'} &= \omega^{\sigma_r} \delta_{\sigma\sigma'} \end{aligned} \tag{6}$$

where

$$\delta_{\sigma\sigma'} = \prod_{i=0}^{m-1} \delta(\sigma_i, \sigma'_i)$$

Then we can write (4) and (5) as

$$\Xi = \text{Trace } \tilde{A}\tilde{B}\tilde{C}\tilde{D} \tag{7}$$

$$\langle \omega^{j\sigma_0} \rangle = \langle Z_0^j \rangle = \Xi^{-1} \text{ Trace } Z_0^j \tilde{A}\tilde{B}\tilde{C}\tilde{D} \tag{8}$$

where $\tilde{A}$ is the N^m by N^m matrix with elements $\tilde{A}_{\sigma\sigma'}$ (taking $\tilde{A}_{\sigma\sigma'}$ to be zero if $\sigma_0 \neq \sigma'_0$, so $\tilde{A}$ commutes with Z_0), and similarly for $\tilde{B}$, $\tilde{C}$, $\tilde{D}$.

These matrices depend on the rapidities p and q. Let the vertical rapidities have value p (p') for lines to the left (right) of center site 0. Similarly, let the lower and uppχr horizontal lines have rapidities q, q', as indicated in Fig. 2. Then, exhibiting the rapidity arguments of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$:

$$\begin{aligned} \tilde{A} &= \tilde{A}_{p'q}, & \tilde{B} &= \tilde{B}_{p'q'} \\ \tilde{C} &= \tilde{C}_{pq'}, & \tilde{D} &= \tilde{D}_{pq} \end{aligned} \tag{9}$$

3. FACTORIZATION PROPERTIES OF THE CTMs

In ref. 7 it was shown that the star–triangle relation imposes severe constraints on the functional form of the corner transfer matrices, but in part that argument depended on the "difference" property, which no longer applies. Here we therefore rederive the appropriate relations. From now on we implicitly consider the phase when the system is ferromagnetically ordered, and when (for all choices p, p' of p, and q, q' of q)

$$|W_{pq}(0)| > |W_{pq}(n)|, \qquad |\bar{W}_{pq}(0)| > |\bar{W}_{pq}(n)|$$

provided $n \neq 0$, modulo N.

That such a phase exists can be seen by taking the "low-temperature" limit, when $k' \to 0$, y_p, x_q, $y_q \to 1$, $\lambda_p = \mathcal{O}(k')$, $\lambda_q = \mathcal{O}(1)$; $x_p = t_p$ and λ_q being finite and arbitrary. Then $W_{pq}(n) = \bar{W}_{pq}(n) = \delta_{n0}$, so the system is completely ferromagnetically ordered, all spins having value 0. (This particular value being determined by the boundary conditions: to consider other phases, possibly inhomogeneous, the boundary conditions should be modified appropriately.)

The various elements of the matrix product $\tilde{B}\tilde{C}$ are boundary correlations of a model defined on the upper half of $\mathscr{L}$. If this had been given cylindrical boundary conditions, such correlations would (in the large-lattice limit) be independent of the horizontal rapidity q'. (They depend only on the elements of the maximal eigenvector of the row-to-row transfer matrix, which is independent of q'.) However, we expect such properties to be independent of boundary conditions in the large-lattice limit, so

$$(\tilde{B}_{p'q'}\tilde{C}_{pq'})_{\sigma'\sigma'''} = \text{independent of } q' \tag{10}$$

This equation is to be interpreted in the limit when $m \to \infty$, while $\sigma'_r, \sigma'''_r = 0$ for $r > r_0$; r_0 being some number independent of m.

In this limit $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are infinite-dimensional matrices. Assuming (as appears to be the case) that we can take them to be invertible, and can regard (10) as elements of a normal matrix equation, it follows that there exist matrices $A_{p'}$, $B_{q'}$, C_p, each a function of only one rapidity, such that

$$\tilde{B}_{p'q'} = A_{p'}^{-1}B_{q'}, \qquad \tilde{C}_{pq'} = B_{q'}^{-1}C_p \tag{11}$$

[The first equation can be obtained from (10) by fixing p to some given value and solving for $\tilde{B}_{p'q'}$. The second then follows.]

Similar product relations and factorizations apply to the other quadrants. We can obtain all of these more explicitly from (11) by using the rotation symmetry of the chiral Potts model.[1,9] Let R be the automorphism such that $x_{Rp} = y_p$, $y_{Rp} = \omega x_p$, $\mu_{Rp} = \mu_p^{-1}$. Then

$$W_{q,Rp}(n) = \bar{W}_{pq}(n), \qquad \bar{W}_{q,Rp} = W_{pq}(-n) \tag{12}$$

It follows that

$$\begin{aligned} \tilde{A}_{q,Rp} &= \tilde{B}_{pq}, \qquad & \tilde{B}_{q,Rp} &= \tilde{C}_{pq} \\ \tilde{C}_{q,Rp} &= \tilde{D}_{pq}, \qquad & \tilde{D}_{q,Rp} &= \tilde{A}_{pq} \end{aligned} \tag{13}$$

Also, when $p = q$, the weights are $W_{pp}(n) = 1$, $\bar{W}_{pp}(n) = \delta_{n0}$, so $\tilde{B}_{pp} = \tilde{D}_{pp} = \mathbf{1}$. It follows that $B_p = A_p$ and

$$\begin{aligned} \tilde{A}_{pq} &= A_{R^{-1}q}^{-1} A_p, \qquad & \tilde{B}_{pq} &= A_p^{-1} A_q \\ \tilde{C}_{pq} &= A_q^{-1} A_{Rp}, \qquad & \tilde{D}_{pq} &= A_{Rp}^{-1} A_{Rq} \end{aligned} \tag{14}$$

where

$$A_{R^2 q} = M A_q, \qquad \forall q \tag{15}$$

M being some rapidity-independent matrix. Hence, from (7) and (8),

$$\begin{aligned} \Xi &= \text{Trace } A_{Rq} A_{R^{-1}q}^{-1} = \text{Trace } M \\ \langle Z_0^j \rangle &= \text{Trace } Z_0^j M / \text{Trace } M \end{aligned} \tag{16}$$

Note that this last result is independent of the normalization of M. It also shows that $\langle Z_0^j \rangle$ is independent of the rapidities p and q, as is required by Z invariance.[10]

The chiral Potts model also has a reflection symmetry.[1,9] If S is the automorphism such that $x_{Sp} = 1/y_p$, $y_{Sp} = 1/x_p$, $\mu_{Sp} = \omega^{-1/2} \mu_p / \mu_p x_p$, then $S^2 = (RS)^2 = \mathbf{1}$ and

$$W_{Sq,Sp}(n) = W_{pq}(n), \qquad \bar{W}_{Sq,Sp}(n) = \bar{W}_{pq}(-n)$$

Hence

$$\begin{aligned} \tilde{A}_{Sq,Sp} &= \tilde{C}_{pq}^T, \qquad & \tilde{B}_{Sq,Sp} &= \tilde{B}_{pq}^T \\ \tilde{C}_{Sq,Sp} &= \tilde{A}_{pq}^T, \qquad & \tilde{D}_{Sq,Sp} &= \tilde{D}_{pq}^T \end{aligned} \tag{17}$$

Substituting the forms (14), we find that these symmetries are equivalent to

$$A_{Sq} A_q^T = Y, \qquad \forall q \tag{18}$$

where Y and MY are symmetric rapidity-independent matrices.

These equations do not define A_p, M, Y uniquely: the corner transfer matrices are unchanged by the transformation $A_p \to LA_p$, $M \to LML^{-1}$, $Y \to LYL^T$, for any invertible rapidity-independent matrix L.

3.1. The Physical Regime

We can define four further "p-variables" u_p, v_p, ϕ_p, $\bar{\phi}_p$, related by

$$\begin{aligned} \sin v_p &= k \sin u_p, & \phi_p &= -v_p \\ k' \cos \bar{\phi}_p &= \cos v_p, & k' \sin \bar{\phi}_p &= -ik \cos u_p \end{aligned} \tag{19}$$

Then we can express $x_p, \ldots, \mu_p$ in terms of these:

$$\begin{aligned} x_p &= \exp[i(u_p - v_p)/N], & y_p &= \omega^{1/2} \exp[i(u_p + v_p)/N] \\ t_p &= \omega^{1/2} \exp(2iu_p/N), & \mu_p &= \exp[i(\bar{\phi}_p + v_p)/N] \end{aligned} \tag{20}$$

and one has the corollaries

$$\begin{gathered} \lambda_p - \lambda_p^{-1} = (2k/k')\, e^{iu_p} \cos v_p \\ \exp(2i\phi_p/N) = \omega^{1/2} x_p / y_p, \qquad \exp(2i\bar{\phi}_p/N) = \omega^{1/2} \mu_p^2 x_p / \mu_p \end{gathered} \tag{21}$$

(The variables u_p and v_p are those used in ref. 9; ϕ_p and $\bar{\phi}_p$ are the ϕ and $\bar{\phi}$ used in ref. 8.)

If u_p is real, there is a unique solution of (19) such that v_p is real, with $-\pi/2 < v_p < \pi/2$, and $\bar{\phi}_p$ is pure imaginary. Then

$$u_{Rp} = -u_{Sp} = u_p + \pi, \qquad v_{Rp} = -v_{Sp} = -v_p, \qquad \bar{\phi}_{Rp} = \bar{\phi}_{Sp} = -\bar{\phi}_p$$

If u_q, v_q, $i\bar{\phi}_q$ are similarly chosen real, and if $0 < u_q - u_p < \pi$, then it was noted in ref. 9 that the Boltzmann weights W, $\bar{W}$ are real and positive. Thus the corner transfer matrices are then real (with nonnegative entries). The matrix A_p can be chosen to be real, so M and Y are then real. Since M and Y are rapidity-independent, they remain real even if u_p and u_q are allowed to move off the real axis into the complex plane.

We can regard A_p as a matrix function $A(u_p)$ of the variable u_p. Then (15) becomes

$$A(u + 2\pi) = MA(u), \qquad \forall u \tag{22}$$

This can be thought of as a "quasiperiodicity" condition on $A(u)$.

3.2. Variation around a Particular Contour C_1

Regarding v_p, $\bar{\phi}_p$ as functions of the complex variable u_p, we have that v_p has branch points at $u_p = (2n-1)\pi/2 \pm i\gamma$ (for all integers n), and $\bar{\phi}_p$ also has branch points at $u_p = \pm i\infty$. They are analytic, in particular single-valued, in the cut plane shown in Fig. 3.

Suppose, however, that one starts u_p from the point P shown in the figure and moves it around the contour C_1 *enclosing* he branch points $-\pi/2 \pm i\gamma$, varying v_p and $\bar{\phi}_p$ continuously. On crossing the lower branch cut, one moves to another Reimann sheet for v_p, but returns to the original on crossing the upper cut, so v_p returns to its original value. So therefore does $\exp(i\bar{\phi}_p)$, but $\bar{\phi}_p$ itself is incremented by (-2π). This means that x_p, y_p return to their original values, but μ_p goes to $\omega^{-1}\mu_p$; i.e., p is replaced by Vp, where V is the automorphism such that $x_{Vp} = x_p$, $y_{Vp} = y_p$, $\mu_{Vp} = \omega^{-1}\mu_p$.

Consider the corner transfer matrix $\tilde{B}_{pq}$ when $q = Vp$. Then $W_{pq}(n) = \omega^n$, $\bar{W}_{pq}(n) = \delta_{n0}$, so $\tilde{B}_{pq}$ is a diagonal matrix, with elements

$$W_{pq}(\sigma_0 - \sigma_1)\, W_{pq}(\sigma_1 - \sigma_2)^3\, W_{pq}(\sigma_2 - \sigma_3)^5 \cdots$$

and

$$\tilde{B}_{p,Vp} = \Omega \tag{23}$$

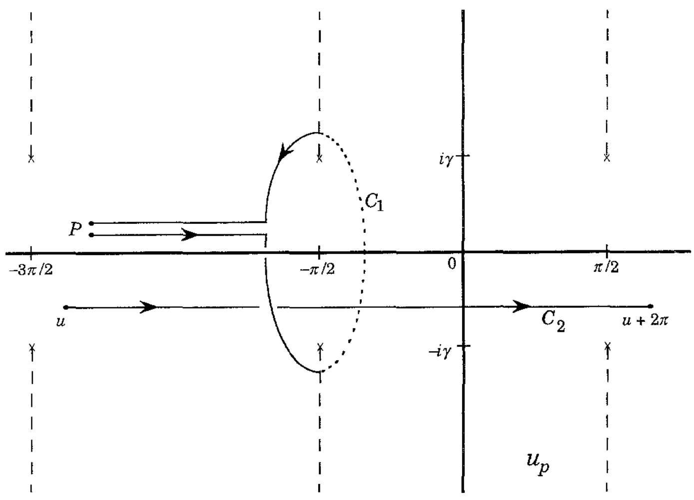

Fig. 3. The complex u_p plane: the branch cuts are shown as broken lines, e.g., from $-\pi/2 + i\gamma$ to $+i\infty$, where $\gamma = \text{arcosh}(1/k)$. Also shown are the two contours C_1 and C_2.

where Ω is the diagonal matrix with elements

$$(\Omega)_{\sigma\sigma'} = \omega^{\sigma_0 + 2\sigma_1 + 2\sigma_2 + 2\sigma_3 + \cdots} \prod_{i=0}^{m-1} \delta(\sigma_i, \sigma_i') \tag{24}$$

Using (14), it follows that

$$A_{Vp} = A_p \Omega \tag{25}$$

We can think of (15) and (25) as "quasiperiodicity" conditions on the matrix function A_p.

4. THE BOOST OPERATOR $\mathscr{B}_p$

We remarked above that $\tilde{B}_{pq} = \mathbf{1}$ when $q = p$. Now consider the case when q is close to p, and expand to first order in the difference. From (2) and (3), we can verify that

$$\begin{aligned} W_{pq}(n) &= 1 + \varepsilon_{pq} \beta_p(n) \\ \bar{W}_{pq}(n) &= \delta_{n0} + \varepsilon_{pq} \bar{\alpha}_p(n) \end{aligned} \tag{26}$$

where

$$\begin{aligned} \varepsilon_{pq} &= (u_q - u_p)/(2N \cos v_p) \\ \beta_p(n) &= \sum_{j=1}^{N-1} \alpha_p(j)[\omega^{jn} - 1] \\ \alpha_p(n) &= \exp[i(2n - N)\, \phi_p/N]/\sin(\pi n/N) \\ \bar{\alpha}_p(n) &= k' \exp[i(2n - N)\, \bar{\phi}_p/N]/\sin(\pi n/N) \end{aligned} \tag{27}$$

Here we are using notation similar to that of Albertini *et al.*[8]: k', $\alpha_p(j)$, $\bar{\alpha}_p(n)$, ϕ_p, $\bar{\phi}_p$ being their λ, α_j, $\bar{\alpha}_n$, ϕ, $\bar{\phi}$. In the physical regime $\bar{\alpha}_p(n)$ is real and positive; $\beta_p(n)$ is real, and appears to be nonpositive.

As in ref. 7, one can expand $\tilde{B}_{pq}$ to first order in ε_{pq}. Using (26), we obtain

$$\tilde{B}_{pq} = A_p^{-1} A_q = \exp(-\varepsilon_{pq} \mathscr{B}_p) \tag{28}$$

where, using the elementary operators X_j, Z_j defined in (6),

$$\mathscr{B}_p = -\sum_{j=1}^{\infty} \sum_{n=1}^{N-1} \{2j\bar{\alpha}_p(n)\, X_j^n + (2j-1)\, \alpha_p(n)[Z_{j-1}^n Z_j^{-n} - \mathbf{1}]\} \tag{29}$$

This $\mathscr{B}_p$ is the "boost operator."[11] Apart from boundary conditions, it differs from the chiral Potts Hamiltonian[8] $\mathscr{H}$ only by the integer coefficients $2j$ and $2j-1$.

We can regard $\mathscr{B}_p$ as known and (28) as a matrix differential equation for A_p. We can formally integrate it as follows.

Let C be some open contour in the complex u_p plane, with endpoints a and b. To each point u_p on C assign values of the other p-variables v_p, ϕ_p, $\bar{\phi}_p$, consistent with (19), so that each varies continuously on C. Choose a sequence (ordered along the contour) of points $u_1,..., u_{n+1}$ on C such that $u_1 = a$ and $u_{n+1} = b$. Let pj denote all the rapidity variables u_p, v_p, ϕ_p, $\bar{\phi}_p$ at the point u_j, the endpoint rapidities being $p(a) = p1$ and $p(b) = p(n+1)$. Let E_{pj} be the matrix

$$E_{pj} = 1 - (u_{j+1} - u_j)\, \mathscr{B}_{pj}/(2N \cos v_{pj}) \tag{30}$$

(note that it depends on $u_{j+1} - u_j$, as well as pj); and define

$$\{E_{p(a)} \cdots E_{p(b)}\}_C = \lim_{n \to \infty} E_{p1} E_{p2} \cdots E_{pn} \tag{31}$$

the limit being taken so that the points $u_1,..., u_{N+1}$ are densely spaced on C, i.e., $|u_{j+1} - u_j| \to 0$, $\forall j$. Then, using (28),

$$A_q = A_p \{E_p \cdots E_q\}_C \tag{32}$$

C being an appropriate contour from u_p to u_q.

Taking C to be the contour C_1, it follows from the above remarks and in particular from (23) that

$$\{E_p \cdots E_{Vp}\}_{C_1} = \Omega \tag{33}$$

Also, if C_2 is the horizontal straight line from u_p to $u_p + 2\pi$, from (15) we have

$$\{E_p \cdots E_{R^2 p}\}_{C_2} = A_p^{-1} M A_p \tag{34}$$

We can think of (33) and (34) as integral equations for $\mathscr{B}_p$, A_p, and M. It should be remembered that these are infinite-dimensional matrices, and some care should be taken to stay in a domain in the u_p plane wherein the (appropriately normalized) matrices exist and the summations implicit in matrix multiplication are convergent. We expect there to be no difficulty for u_p real, since this case arises in the physical (ferromagnetic) regime. From low-temperature expansions, it appears that we can extend off the real axis, in particular as far as the contour C_1. In (34), u_p should be confined to the horizontal strip $|\mathrm{Im}(u_p)| < \gamma$, so that C_2 does not cross any of the branch cuts in Fig. 3.

Our ultimate aim (not yet achieved) is to use (33) and (34) to obtain the diagonalized form of M (to within an overall normalization factor), and then to calculate the center-spin averages ("magnetizations") from (16). Albertini *et al.* have conjectured (ref. 8, §4) (on the basis of series expansions[12,13] and the result for the $N=2$ Ising case) that

$$\langle Z_0^j \rangle = (1-k'^2)^{j(N-j)/2N^2} \tag{35}$$

We should like to verify this.

5. $N=2$: THE ISING CASE

It is very simple to carry out this program when $N=2$. Then $\alpha_p(1)=1$, $\bar{\alpha}_p(1)=k'$, so $\mathscr{B}_p=\mathscr{B}$ is a constant matrix, independent of p. Hence

$$\{E_p \cdots E_q\}_C = \exp(-s_{pq}\mathscr{B})$$

where

$$s_{pq} = \int_{u_p}^{u_q} \frac{du}{4(1-k^2\sin^2 u)^{1/2}}$$

the integration being along the contour C. If K, K' are the usual complete elliptic integrals of the first kind (ref. 14, §8.112), then

$$s_{pq} = -iK' \qquad \text{for} \quad C=C_1$$

$$s_{pq} = K \qquad \text{for} \quad C=C_2$$

From (33) we therefore have

$$\exp(iK'\mathscr{B}) = \Omega \tag{36}$$

The matrix Ω commutes with $\mathscr{B}$ and has eigenvalues ± 1. Considering (36) in its diagonal representation, it follows that the eigenvalues of $K'\mathscr{B}/\pi$ must all be integers. By continuity, they must therefore be independent of k'. If $\mathscr{B}_{\mathrm{diag}}$ is the diagonalized form of $\mathscr{B}$, it follows that $K'\mathscr{B}_{\mathrm{diag}}$ is independent of k'. We can evaluate it by taking the limit $k' \to 0$ in (29), when $\bar{\alpha}_p(1) \to 0$ and the RHS becomes diagonal. Hence, for $0 < k' < 1$,

$$\mathscr{B}_{\mathrm{diag}} = -(\pi/2K') \sum_{j=1}^{\infty} (2j-1)[Z_{j-1}Z_j^{-1} - \mathbf{1}] \tag{37}$$

From (34), the diagonalized form of M is therefore

$$M = \exp(-K\mathscr{B}_{\text{diag}}) \tag{38}$$

Introducing "edge spins" $\mu_1, \mu_2,\ldots$, related to the site spins $\sigma_0, \sigma_1,\ldots$ by $\mu_j = \sigma_{j-1}\sigma_j$, and expressing (37) in terms of the edge spins, it follows that

$$M = \begin{pmatrix} 1 & 0 \\ 0 & q' \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & q'^3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & q'^5 \end{pmatrix} \otimes \cdots$$

$$Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots$$

where

$$q' = e^{-\pi K/K'}$$

is the "nome" of the modulus k'. Substituting these forms into (16) and using ref. 14, §8.197.4, we find that the spontaneous magnetization of the Ising model is

$$\langle Z_0 \rangle = \prod_{j=1}^{\infty} \left(\frac{1 - q'^{2j-1}}{1 + q'^{2j-1}} \right) = k^{1/4} = (1 - k'^2)^{1/8} \tag{39}$$

in agreement with Onsager[15] and Yang.[16]

6. THE LOW-TEMPERATURE LIMIT

Return to the general-N case, and let $k' \to 0$, keeping u_p fixed. The branch cuts in Fig. 3 close in on the real axis, dividing the u_p plane into vertical strips of width π. Depending on which strip u_p is in, $\exp(i\bar{\phi}_p)$ becomes of order k', $1/k'$, or 1. (The last case arises when u_p is close to a pair of branch points.) In any event, $\bar{\alpha}_p(n) \to 0$ at least as fast as $k'^{2/N}$, for $1 \leqslant n \leqslant N-1$. On the other hand, $\alpha_p(n)$ tends to a nonzero limit, so $\mathscr{B}_p$ becomes diagonal. The definition (31) then simplifies to

$$\{E_p \cdots E_q\}_C = \exp\left\{ \sum_{j=1}^{\infty} (2j-1) \sum_{n=1}^{N-1} G(n; C)[Z_{j-1}^n Z_j^{-n} - \mathbf{1}]/(2N) \right\} \tag{40}$$

where

$$G(n; C) = \int_C \frac{\alpha(n)\, du}{\cos v} \tag{41}$$

Here u is a variable of integration that follows the contour C in the complex u plane from u_p to u_q; v is related to u by

$$\sin v = k \sin u \tag{42}$$

and $\alpha(n)$ is the function $\alpha_p(n)$ of (27), with $\phi_p = -v_p$ replaced by $-v$. Changing the variable of integration from u to v, it follows that

$$G(n; C) = \frac{1}{\sin(\pi n/N)} \int_{C_v} \frac{e^{i(N-2n)v/N}\, dv}{k \cos u} \tag{43}$$

where C_v is the corresponding contour in the v plane, going from v_p to v_q.

The corner transfer matrix $\tilde{B}_{pq} = A_p^{-1} A_q = \{E_p \cdots E_q\}_C$ is therefore diagonal, with elements

$$(A_p^{-1} A_q)_{\sigma\sigma'} = [h(\sigma_0 - \sigma_1)\, h(\sigma_1 - \sigma_2)^3\, h(\sigma_2 - \sigma_3)^5 \cdots]\, \delta(\sigma, \sigma') \tag{44}$$

where $h(0)$, $h(1)$,... depend implicitly on p, q, and C, and are given by

$$h(\mu) = \exp\left\{\sum_{n=1}^{N-1} G(n; C)(\omega^{n\mu} - 1)/2N\right\} \tag{45}$$

As in the previous section, we can express this result in terms of "edge spins" $\mu_1, \mu_2,\ldots$, related now to the site spins $\sigma_0, \sigma_1,\ldots$ by $\mu_j = \sigma_{j-1} - \sigma_j$ (mod N). Then (44) can be written as

$$A_p^{-1} A_q = H_{pq} \otimes H_{pq}^3 \otimes H_{pq}^5 \otimes \cdots \tag{46}$$

where H_{pq} is the N by N diagonal matrix

$$H_{pq} = \begin{pmatrix} h(0) & 0 & \cdots & 0 \\ 0 & h(1) & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & \cdot & \cdots & h(N-1) \end{pmatrix} \tag{47}$$

First consider the case when $C = C_1$. When u is moved around the contour C_1 in Fig. 3, the corresponding contour C_v in the v plane can be shrunk to just surround the segment $(\theta - \pi, -\theta)$ of the real axis ($v = \theta - \pi$ and $-\theta$ are branch points of $\cos u$, considered as a function of v). On C_v, $dv/\cos u$ is negative imaginary, so

$$G(n; C_1) = \frac{-2i}{\sin(\pi n/N)} \int_{\theta-\pi}^{-\theta} \frac{e^{i(N-2n)v/N}\, dv}{(\sin^2 v - \sin^2 \theta)^{1/2}} \tag{48}$$

where

$$k = \sin\theta, \qquad k' = \cos\theta, \qquad 0 < \theta < \pi/2 \tag{49}$$

From the Appendix (replacing θ, y therein by $\pi/2 - \theta$, $v - \pi/2$) we obtain

$$\begin{aligned} G(n; C_1) &= -\frac{2e^{in\pi/N}}{\sin(\pi n/N)} I\left(\frac{\pi}{2} - \theta, \frac{n}{N}\right) \\ &= \frac{4\pi i}{\omega^{-n} - 1} F\left(\frac{n}{N}, \frac{N-n}{N}; 1; k'^2\right) \end{aligned} \tag{50}$$

When $k' \to 0$, the function F in (50) tends to 1, so from (45), for all integers μ,

$$\begin{aligned} h(\mu) &= \exp\left(\frac{2\pi i}{N} \sum_{n=1}^{N-1} \frac{\omega^{n\mu} - 1}{\omega^{-n} - 1}\right) \\ &= \omega^{\mu} \end{aligned} \tag{51}$$

Hence (44) gives

$$(A_p^{-1} A_q)_{\sigma\sigma'} = \omega^{\sigma_0 + 2\sigma_1 + 2\sigma_2 + \cdots} \tag{52}$$

in agreement with (25).

Now take $C = C_2$. Then the corresponding contour C_v just surrounds the segment $(-\theta, \theta)$ of the real axis, in the positive (anticlockwise) direction; and $dv/\cos u$ is positive real on C_v. Hence, from (43),

$$G(n; C_2) = \frac{2}{\sin(\pi n/N)} \int_{-\theta}^{\theta} \frac{e^{i(N-2n)v/N}\, dv}{(\sin^2\theta - \sin^2 v)^{1/2}} \tag{53}$$

Again using (A1)–(A7) of the Appendix, we get

$$G(n; C_2) = \frac{2\pi}{\sin(\pi n/N)} F\left(\frac{n}{N}, \frac{N-n}{N}; 1; k^2\right) \tag{54}$$

For z close to 1, by taking the limit $\gamma \to \alpha + \beta$ in ref. 14, §9.131.2, we can verify that

$$\begin{aligned} F(\alpha, 1-\alpha; 1; z) = &-\frac{\sin \pi\alpha}{\pi} \{\ln(1-z) + \psi(\alpha) + \psi(1-\alpha) - 2\psi(1)\} \\ &+ \mathcal{O}\{(1-z)\ln(1-z)\} \end{aligned} \tag{55}$$

where $\psi(x) = d \ln \Gamma(x)/dx$ is Euler's psi function.

From ref. 14, §8.363.6, inserting an omitted term $-\ln 2$ in the RHS, for $n = 1,\ldots, N-1$,

$$\psi\left(\frac{n}{N}\right) = -C - \ln 2N - \frac{\pi}{2}\cot\frac{\pi n}{N} + \sum_{j=1}^{N-1}\left[\cos\frac{2jn\pi}{N}\ln\sin\frac{\pi j}{N}\right] \tag{56}$$

$C = -\psi(1) = 0.577215\ldots$ is Euler's constant.

For all integers μ, define

$$\begin{aligned} Q(\mu) &= 1 && \text{if} \quad \mu = 0, \quad \text{mod } N \\ &= k'^2/[4N^2\sin^2(\pi\mu/N)] && \text{otherwise} \end{aligned} \tag{57}$$

Then from (54)–(56) it follows that

$$h(\mu) = Q(\mu) \tag{58}$$

and hence, from (34) and (44), M is the diagonal matrix with elements

$$M_{\sigma\sigma'} = [Q(\sigma_0 - \sigma_1)\, Q(\sigma_1 - \sigma_2)^3\, Q(\sigma_2 - \sigma_3)^5 \cdots]\, \delta_{\sigma\sigma'} \tag{59}$$

To order k'^2, it follows from (16) that

$$\langle Z_0^j \rangle = \sum_{n=0}^{N-1} \omega^{nj} Q(n) \Big/ \sum_{n=0}^{N-1} Q(n) \tag{60}$$

and that $Q(n)/[Q(1) + \cdots + Q(N-1)]$ is the probability that the center spin σ_0 has value n. This can be verified directly: to this order we can take all the other spins on the lattice to be zero, so the probability is proportional to

$$\mathscr{P}(n) = W_{pq}(n)\, W_{pq}(N-n)\, \overline{W}_{pq}(n)\, \overline{W}_{pq}(N-n)$$

Taking the low-temperature limit as in the second paragraph of Section 3, we find

$$W_{pq}(n)\, \overline{W}_{pq}(N-n) = k'\mu_q^{N-2n}/[N(1-\omega^n)]$$

and hence $\mathscr{P}(n) = Q(n)$.

Using the identity, true for $0 \leqslant j \leqslant N$,

$$\sum_{n=1}^{N-1} \frac{\omega^n(\omega^{nj} - 1)}{(1-\omega^n)^2} = \frac{j(N-j)}{2} \tag{61}$$

we find from (60) and (57) that

$$\langle Z_0^j \rangle = 1 - j(N-j)\, k'^2/2N^2 + \mathcal{O}(k'^4) \tag{62}$$

in agreement with the conjecture (35) of Albertini *et al.*[8]

6.1. Formulation Using Hyperelliptic Variables

In this section we are concerned with the low-temperature limit, when $k' \to 0$. However, for all k', (43) is a hyperelliptic integral of *precisely* the type that provides a uniformizing substitution for the relation (42) (considered as a relation between e^{2iu} and $e^{2iv/N}$). These integrals and associated functions were used by Kowalevski in her classic work[17] on the rotation of a rigid body: in ref. 4 we have adapted and specialized some of her formulas to this case, and shown that they provide a uniformizing parametrization of x_p, y_p, μ_p.

In particular, as in (5) and (35) of ref. 4 (with u therein replaced by $-u-\pi$), we take

$$\Delta(v) = ik \cos u = \{\sin^2 v - \sin^2 \theta\}^{1/2} \tag{63}$$

$$w_n = \int_\theta^{v_p} \frac{e^{i(N-2n)v/N}\, dv}{\Delta(v)}, \qquad n = 1,\ldots, N-1 \tag{64}$$

Define w_n' similarly, but with v_p replaced by v_q: thus $w_1,\ldots, w_{N-1}$ are functions of the rapidity p; $w_1',\ldots, w_{N-1}'$ of the rapidity q. Then (43) simplifies to

$$G(n; C) = \frac{i(w_n' - w_n)}{\sin(\pi n/N)} \tag{65}$$

We are choosing the signs appropriately for the physical regime wherein $-3\pi/2 < u_p < u_q < -\pi/2$, $-\theta < v_q < v_p < \theta$: for other values they should be taken to be those of the appropriate analytic continuation.

As in ref. 4, for $\alpha, \beta = 1,\ldots, N-1$, define

$$L_\alpha = \int_{-\theta}^{\theta} \frac{e^{i(N-2\alpha)y/N}\, dy}{\{\cos^2\theta - \sin^2 y\}^{1/2}} \tag{66}$$

$$L_\alpha' = \int_{-\theta}^{\theta} \frac{e^{i(N-2\alpha)y/N}\, dy}{\{\sin^2\theta - \sin^2 y\}^{1/2}} \tag{67}$$

$$K_{\alpha\beta} = i\omega^{-\alpha\beta} e^{\pi i\alpha/N} L_\alpha, \qquad \bar{K}_{\alpha\beta} = \omega^{\alpha-\alpha\beta} L_\alpha' \tag{68}$$

$$K_{\alpha\beta}' = \sum_{\gamma=1}^{\beta} \bar{K}_{\alpha\gamma} \tag{69}$$

and define quantities $\tau_{\alpha\beta}$ by

$$K_{\alpha\beta}' = -i \sum_{\gamma=1}^{N-1} K_{\alpha\gamma} \tau_{\gamma\beta} \tag{70}$$

Then

$$\tau_{\alpha\beta} = \rho_\alpha + \rho_\beta - \rho_{\alpha-\beta} \tag{71}$$

where

$$\rho_\alpha = \rho_{-\alpha} = \rho_{N-\alpha} = \frac{i}{N} \sum_{\gamma=1}^{N-1} \frac{\sin^2(\pi\alpha\gamma/N)\, L'_\gamma}{\sin(\pi\gamma/N)\, L_\gamma} \tag{72}$$

Also, from the Appendix,

$$\begin{aligned} L_\alpha &= \pi F\left(\frac{\alpha}{N}, \frac{N-\alpha}{N}; 1; \cos^2\theta\right) \\ L'_\alpha &= \pi F\left(\frac{\alpha}{N}, \frac{N-\alpha}{N}; 1; \sin^2\theta\right) \end{aligned} \tag{73}$$

while (48) and (53) can be written

$$G(n; C_1) = -2e^{\pi i n/N} L_n / \sin(\pi n/N) \tag{74}$$

$$G(n; C_2) = 2L'_n / \sin(\pi n/N) \tag{75}$$

To obtain the desired symmetry property (25), we had to note that $L_n = \pi$ when $k' = 0$. Alternatively, and it turns out very naturally, we can replace $G(n; C)$ in (40) and (45) by

$$\hat{G}(n; C) = \frac{\pi G(n; C)}{L_n} = \frac{\pi i(w'_n - w_n)}{\sin(\pi n/N)\, L_n} \tag{76}$$

Then (44) satisfies (25) exactly, for *all* k'.

Define $s_1, \ldots, s_{N-1}$ in terms of $w_1, \ldots, w_{N-1}$ by Eq. (19) of ref. 4, i.e.,

$$w_\alpha = 2 \sum_{\beta=1}^{N-1} K_{\alpha\beta} s_\beta, \qquad \alpha = 1, \ldots, N-1 \tag{77}$$

Define $s'_1, \ldots, s'_{N-1}$ similarly in terms of $w'_1, \ldots, w'_{N-1}$. Then, for $\mu = 0, 1, \ldots, N$, (45) gives

$$\begin{aligned} \ln h(\mu) &= \sum_{n=1}^{N-1} \frac{\hat{G}(n; C)(\omega^{n\mu} - 1)}{2N} \\ &= \frac{\pi i}{2N} \sum_{n=1}^{N-1} \frac{(\omega^{n\mu} - 1)(w'_n - w_n)}{\sin(\pi n/N)\, L_n} \\ &= \frac{2\pi i \mu}{N} \sum_{\beta=1}^{N-1} (s'_\beta - s_\beta) - 2\pi i \sum_{\beta=1}^{\mu} (s'_\beta - s_\beta) \end{aligned} \tag{78}$$

Setting

$$z_\beta = e^{2\pi i s_\beta}, \qquad z'_\beta = e^{2\pi i s'_\beta} \tag{79}$$

we obviously have

$$h(\mu) = \frac{z_1 \cdots z_\mu}{z'_1 \cdots z'_\mu} \left\{ \frac{z'_1 \cdots z'_{N-1}}{z_1 \cdots z_{N-1}} \right\}^{\mu/N} \tag{80}$$

Thus, at least at low temperatures, the corner transfer matrices take a very simple form when appropriately expressed in terms of the hyperelliptic variables $s_1,..., s_{N-1}$, $s'_1,..., s'_{N-1}$. Note in particular that s_β and s'_β, which correspond to the two rapidities p and q, enter (78) only via their difference: to this extent we have regained the "difference property" that is lacking in the chiral Potts model.[1]

From (65) and (74), w'_n is incremented by $2K_{n0}$ when u is moved around the contour C_1: then, from (68) and (77), $s'_1,..., s'_{N-1}$ are each decreased by unity and it is obvious from (78) that $h(\mu)$ is multiplied by ω^μ.

When u is moved around the contour C_2, each w'_n is incremented by $-2iK'_{n1}$, so from (70), each s'_α is incremented by $-\tau_{\alpha 1}$. From (78), the increment in $\ln h(\mu)$ is therefore

$$-(2\pi i\mu/N) \sum_{\beta=1}^{N-1} \tau_{\beta 1} + 2\pi i \sum_{\beta=1}^{\mu} \tau_{\beta 1} \tag{81}$$

Using (71), we find this is simply $2\pi i \rho_\mu$, so $h(\mu)$ is multiplied by $\exp(2\pi i \rho_\mu)$, and to leading order in a small-k' expansion we must have

$$Q(\mu) = \exp(2\pi i \rho_\mu) \tag{82}$$

Using (72), we can rederive (57)–(58): this is a simpler method than using the cumbersome formulas (55), (56).

6.2. A Wrong Conjecture

Guided by the form of the relation (15) and the Ising, eight-vertex, and other models (Chapters 13 and 14 of ref. 7), it is natural to hope that the eigenvalues of M are (to within an overall normalization factor) products of integer powers of the $Q(1),..., Q(N-1)$ defined by (82). If so, and if we are correct in neglecting the $\bar{\alpha}_p(n)$ when deriving (40) in the limit $k' \to 0$, then by continuity (59) should give the exact diagonalized form of

M for all k' between 0 and 1. For $N=2$ this is true, but for $N \geqslant 3$ it appears to fail. In particular, for $N=3$ it gives, using (16),

$$\langle Z_0^1 \rangle = \langle Z_0^2 \rangle = \frac{1-q}{1+2q} \frac{1-q^3}{1+2q^3} \frac{1-q^5}{1+2q^5} \cdots \tag{83}$$

where $q = Q(1) = Q(2) = \exp(2\pi i \rho_1)$.

We can take q to be defined by Eq. (A9) of ref. 4, i.e.,

$$\left(\frac{k'}{k}\right)^{1/3} = 3^{1/2} q^{1/6} \prod_{n=1}^{\infty} \left(\frac{1-q^{3n}}{1-q^n}\right)^2 \tag{84}$$

so

$$q = \frac{k'^2}{27} + \frac{5k'^4}{243} + \mathscr{O}(k'^6) \tag{85}$$

and (83) gives

$$\begin{aligned} \langle Z_0^1 \rangle = \langle Z_0^2 \rangle &= 1 - 3q + 6q^2 + \cdots \\ &= 1 - \frac{k'^2}{9} - \frac{13k'^4}{243} - \cdots \end{aligned} \tag{86}$$

However, this disagrees with the conjecture (35) which is correct[12] to order $k'^{(13)}$, and gives

$$\langle Z_0^1 \rangle = \langle Z_0^2 \rangle = 1 - 3q + 9q^2 + \cdots = 1 - \frac{k'^2}{9} - \frac{4k'^4}{81} - \cdots \tag{87}$$

Thus these simple ideas do not correctly give the eigenvalues of the CTM product matrix M. We need some further insight to obtain them.

The corresponding results for previous models have led to a rich mathematical theory linking solvable statistical mechanical models to Lie algebras.[18] One would hope that the chiral Potts model will ultimately further extend this theory.

NOTE ADDED IN PROOF

Series expansions (which will be reported elsewhere) give results consistent with the factorization properties (14)–(16). For $N=3$, normalizing M so that its largest eigenvalue is unity, its 15 largest eigenvalues are 1, τ, τ, $2\tau^3 - \frac{1}{2}\tau^4 + \cdots$, $2\tau^3 - \frac{1}{2}\tau^4 + \cdots$, $(\sqrt{2}+1)^2\tau^4$, $(\sqrt{2}-1)^2\tau^4$, $\frac{1}{2}\tau^4$, $\frac{1}{2}\tau^4$, $3\tau^5$, $3\tau^5$, $(\sqrt{3}+1)^2\tau^6$, $(\sqrt{3}-1)^2\tau^6$, $4\tau^6/3$, $4\tau^6/3$, where $\tau = q - q^2 + 5q^3 - 21q^4 + \cdots$, q is given by (84), and eigenvalues 6 to 15 are given to leading order only. Subsequent eigenvalues are of order τ^7 or higher. The corresponding first 15 eigenvalues of Z_0 (which commutes with M) are 1, ω, ω^2, ω, ω^2, 1, 1, ω, ω^2, ω, ω^2, 1, 1, ω, ω^2.

These expressions are consistent with the conjecture (35), but reveal that even in the low-temperature limit the diagonalized form of M does *not* have the simple direct product structure of (59). This discrepancy is due to neglecting the off-diagonal elements of $\mathscr{B}_p$, i.e., those involving $\bar{\alpha}_p(n)$, in (40).

APPENDIX

Here we consider the integrals that occur in Section 6 and express them in terms of hypergeometric functions, using the formulas of ref. 14.

Taking θ, α to be real constants, with $0<\theta<\pi/2$, let

$$I(\theta, \alpha)=\int_{-\theta}^{\theta} \frac{e^{i(1-2\alpha) y}\, dy}{\{\sin^2\theta-\sin^2 y\}^{1/2}} \tag{A1}$$

Setting $x=e^{2iy}$ and $e=e^{2i\theta}$, this becomes

$$I(\theta, \alpha)= -i\int_{e^{-1}}^{e} \frac{x^{-\alpha}\, dx}{\{(1-ex)(1-e^{-1}x)\}^{1/2}} \tag{A2}$$

The integration is around an arc of the unit circle (the one including the point $x=1$), but for $0<\theta<\pi/4$ this is equivalent to a straight line from e^{-1} to e. Setting $t=(1-ex)/(1-e^2)$, we then get

$$I(\theta, \alpha)=e^{\alpha}\int_0^1 \frac{[1-(1-e^2)t]^{-\alpha}\, dt}{\{t(1-t)\}^{1/2}} \tag{A3}$$

Using ref. 14, §9.111, noting that $B(1/2, 1/2)=\pi$, we obtain

$$I(\theta, \alpha)=\pi e^{\alpha} F(\alpha, \tfrac{1}{2}; 1; 1-e^2) \tag{A4}$$

where $F(\alpha, \beta; \gamma; z)$ is the usual hypergeometric function.

Using the transformation of ref. 14, §9.134.1, we can write (A4) as

$$I(\theta, \alpha)=\frac{\pi}{(\cos 2\theta)^{\alpha}} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; 1; -\tan^2 2\theta\right) \tag{A5}$$

From ref. 14, §9.131.1, this is

$$I(\theta, \alpha)=\pi F\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1; \sin^2 2\theta\right) \tag{A6}$$

and finally we use the reciprocal of ref. 14, §9.133 (with $z = \sin^2 \theta$) to get

$$I(\theta, \alpha) = \pi F(\alpha, 1 - \alpha; 1; \sin^2 \theta) \tag{A7}$$

This derivation is only valid as written if $0 < \theta < \pi/4$. However, the right-hand sides of both (A1) and (A7) are analytic for $0 < \theta < \pi/2$, so (A7) is true throughout this larger interval.

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